The almost sure limits of the minimal position and the additive martingale in a branching random walk

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Summary. Consider a real-valued branching random walk in the boundary case. Using the techniques developed by Aïdékon and Shi [5], we give two integral tests which describe respectively the lower limits for the minimal position and the upper limits for the associated additive martingale.

1 Introduction

Let $\{V(u), u \in \mathbb{T}\}$ be a discrete-time branching random walk on the real line $\mathbb{R}$, where $\mathbb{T}$ is an Ulam-Harris tree which describes the genealogy of the particles and $V(u) \in \mathbb{R}$ is the position of the particle u. When a particle u is at n-th generation, we write $|u| = n$ for $n \geq 0$. The branching random walk V can be described as follows: At the beginning, there is a single particle $\varnothing$ located at 0. The particle $\varnothing$ is also the root of $\mathbb{T}$. At the generation 1, the root dies and gives birth to some point process $\mathcal{L}$ on $\mathbb{R}$. The point process $\mathcal{L}$ constitutes the first generation of the branching random walk $\{V(u), |u| = 1\}$. The next generations are defined by recurrence: For each $|u| = n$ (if such u exists), the particle u dies at the $(n+1)$-th generation and gives birth to an independent copy of $\mathcal{L}$ shifted by $V(u)$. The collection of all children of all u together with their positions gives the $(n+1)$-th generation. The whole system may survive forever or die out after some generations.

Plainly $\mathcal{L} = \sum_{|u|=1} \delta_{\{V(u)\}}$. Assuming $\mathbb{E}[\mathcal{L}(\mathbb{R})] \in (1, \infty)$ [namely $\mathbb{T}$ is a supercritical Galton-Watson tree] and that

$$\mathbb{E}\left(\int e^{-x}\mathcal{L}(dx)\right) = 1, \qquad \mathbb{E}\left(\int x e^{-x}\mathcal{L}(dx)\right) = 0. \tag{1.1}$$

When the hypothesis (1.1) is fulfilled, the branching random walk is called in the boundary case in the literature (see e.g. Biggins and Kyprianou [9] and [10], Aïdékon and Shi [5]). Under some integrability conditions, a general branching random walk can be reduced to the boundary case after a linear transformation, see Jaffuel [19] for detailed discussions. We shall assume (1.1) throughout this paper.

Denote by $\mathbb{M}_n := \min_{|u|=n} V(u)$ the minimal position of the branching random walk at generation n (with convention $\inf \emptyset \equiv \infty$). Hammersly [17], Kingman [20] and Biggins [8] established the law of large numbers for $\mathbb{M}_n$ (for any general branching random walk), whereas the second order limits have recently attracted many attentions, see [1, 18, 12, 2] and the references therein. In particular, Aïdékon [2] proved the

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convergence in law of $\mathbb{M}_n - \frac{3}{2}\log n$ under (1.1) and some mild conditions, which gives a discrete analog of Bramson [11]'s theorem on the branching brownian motion.

Concerning on the almost sure limits of $\mathbb{M}_n$, there is a phenomena of fluctuation at the logarithmic scale ([18]): Under (1.1) and some extra integrability assumption: $\exists\, \delta > 0$ such that $\mathbb{E}[\mathcal{L}(\mathbb{R})^{1+\delta}] < \infty$ and $\mathbb{E}\big[\int_{\mathbb{R}}(e^{\delta x} + e^{-(1+\delta)x})\mathcal{L}(dx)\big] < \infty$, the following almost sure limits hold:

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{\mathbb{M}_n}{\log n} &= \frac{3}{2}, & \mathbb{P}^*\text{-a.s.}, \\
\liminf_{n\to\infty} \frac{\mathbb{M}_n}{\log n} &= \frac{1}{2}, & \mathbb{P}^*\text{-a.s.},
\end{aligned}
$$

where here and in the sequel,

$$
\mathbb{P}^*(\cdot) := \mathbb{P}\left(\cdot|\mathcal{S}\right),
$$

and $\mathcal{S}$ denotes the event that the whole system survives. The upper bound $\frac{3}{2}\log n$ is the usual fluctuation for $\mathbb{M}_n$ because $\mathbb{M}_n - \frac{3}{2}\log n$ converges in law ([2]). It is a natural question to ask how $\mathbb{M}_n$ can approach the unusual lower bound $\frac{1}{2}\log n$.

Aïdékon and Shi [5] proved that under (1.1) and the following integrability condition

$$
\sigma^2 := \mathbb{E}\Big[\int_{\mathbb{R}} x^2 e^{-x}\mathcal{L}(dx)\Big] < \infty, \tag{1.2}
$$

$$
\mathbb{E}\Big[\eta(\log_+ \eta)^2 + \widetilde{\eta}\log_+ \widetilde{\eta}\Big] < \infty, \tag{1.3}
$$

where $\eta := \int_{\mathbb{R}} e^{-x}\mathcal{L}(dx)$, $\widetilde{\eta} := \int_0^\infty x\, e^{-x}\mathcal{L}(dx)$ and $\log_+ x := \max(0, \log x)$, then

$$
\liminf_{n\to\infty}\left(\mathbb{M}_n - \frac{1}{2}\log n\right) = -\infty, \qquad \mathbb{P}^*\text{-a.s.}
$$

Furthermore, they asked whether there is some deterministic sequence $a_n \to \infty$ such that

$$
-\infty < \liminf_{n\to\infty} \frac{1}{a_n}\left(\mathbb{M}_n - \frac{1}{2}\log n\right) < 0, \qquad \mathbb{P}^*\text{-a.s.?}
$$

The answer is yes: we can choose $a_n = \log\log n$. Moreover, we can give an integral test to describe the lower limits of $\mathbb{M}_n$:

Theorem 1.1 *Under* (1.1)*,* (1.2) *and* (1.3)*. For any function* $f \uparrow \infty$,

$$
\mathbb{P}^*\left(\mathbb{M}_n - \frac{1}{2}\log n < -f(n), \quad i.o.\right) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^\infty \frac{dt}{t\exp(f(t))} \begin{cases} < \infty \\ = \infty \end{cases}, \tag{1.4}
$$

where i.o. means infinitely often as the relevant index $n \to \infty$.

As a consequence of the integral test (1.4), we have that for any $\varepsilon > 0$, $\mathbb{P}^*$-a.s. for all large $n \geq n_0(\omega)$, $\mathbb{M}_n - \frac{1}{2}\log n \geq -(1+\varepsilon)\log\log n$ whereas there exists infinitely often n such that $\mathbb{M}_n - \frac{1}{2}\log n \leq -\log\log n$. Hence $\mathbb{P}^*$-a.s., $\liminf_{n\to\infty}\frac{1}{\log\log n}(\mathbb{M}_n - \frac{1}{2}\log n) = -1$.

The behaviors of the minimal position $\mathbb{M}_n$ are closely related to the so-called additive martingale $(W_n)_{n\geq 0}$:

$$W_n := \sum_{|u|=n} e^{-V(u)}, \qquad n \geq 0,$$

with the usual convention: $\sum_{\emptyset} \equiv 0$. By Biggins [8] and Lyons [23], $W_n \to 0$ almost surely as $n \to \infty$. The problem to find the rate of convergence (or a Seneta-Heyde norming) for W_n was arisen in Biggins and Kyprianou [9] and was studied in [18]. Aïdékon and Shi [5] gave a definite result to this problem. Let

$$D_n := \sum_{|u|=n} V(u)e^{-V(u)}, \qquad n \geq 1, \tag{1.5}$$

be the derivative martingale (which is a martingale under the boundary condition (1.1)). It was shown in Biggins and Kyprianou [9] that $\mathbb{P}$-a.s., D_n converges to some nonnegative random variable D_∞. Moreover under (1.1), (1.2) and (1.3), $\mathbb{P}^*$-a.s., $D_\infty > 0$, as shown in [9] and [2].

Theorem (Aïdékon and Shi [5]). *Under* (1.1), (1.2) *and* (1.3). *Then under* $\mathbb{P}^*$,

$$\sqrt{n}W_n \overset{(p)}{\to} \sqrt{\frac{2}{\pi\sigma^2}}\, D_\infty,$$

as $n \to \infty$. *Moreover*

$$\limsup_{n\to\infty} \sqrt{n}\, W_n = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Furthermore they conjectured that

$$\liminf_{n\to\infty} \sqrt{n}\, W_n = \sqrt{\frac{2}{\pi\sigma^2}}\, D_\infty, \qquad \mathbb{P}^*\text{-a.s.} \tag{1.6}$$

The upper limits of W_n can be described as follows:

Theorem 1.2 *Under* (1.1), (1.2) *and* (1.3). *For any function* $f \uparrow \infty$, $\mathbb{P}^*$-*almost surely,*

$$\limsup_{n\to\infty} \frac{\sqrt{n}\, W_n}{f(n)} = \begin{cases} 0 \\ \infty \end{cases} \iff \int^\infty \frac{dt}{tf(t)} \begin{cases} < \infty \\ = \infty \end{cases}. \tag{1.7}$$

Concerning the lower limits of W_n, we confirm (1.6) under a stronger integrability assumption: There exists some small constant $\varepsilon_0 > 0$ such that

$$\mathbb{E}\left[\eta^{1+\varepsilon_0} + \int e^{-x}|x|^{2+\varepsilon_0}\mathcal{L}(dx)\right] < \infty. \tag{1.8}$$

It is easy to see that the condition (1.8) is stronger than (1.2) and (1.3).

Proposition 1.3 *Under* (1.1) *and* (1.8). *We have*

$$\liminf_{n\to\infty} \sqrt{n}\, W_n = \sqrt{\frac{2}{\pi\sigma^2}}\, D_\infty, \qquad \mathbb{P}^*\text{-a.s.}$$

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Combining Theorems 1.1 and 1.2, we can roughly say that the main contribution to the upper limits of W_n comes from the term $e^{-\mathbb{M}_n}$. According to Madaule [25], and Aïdékon, Berestichi, Brunet and Shi [3], Arguin, Bovier and Kistler [6] (for the branching brownian motion), the branching random walk seen from the minimal position converges in law to some point process, in particular, $W_n e^{\mathbb{M}_n}$ converges in law as $n \to \infty$, but we are not able to determine the almost sure fluctuations of $W_n e^{\mathbb{M}_n}$.

The whole paper uses essentially the techniques developed by Aïdékon and Shi [5]. To show Theorems 1.1 and 1.2, we firstly remark that both two theorems share the same integral test and that $W_n \geq e^{-\mathbb{M}_n}$, then it is enough to prove the convergence part in the integral test (1.7) and the divergence part in (1.4). The convergence part in (1.7) will follow from an application of Doob's maximal inequality to a certain martingale. To prove the divergence part in (1.4), we shall use the arguments in Aïdékon and Shi [5] (the proof of their Lemma 6.3) to estimate a second moment, then apply Borel-Cantelli's lemma. We can also directly prove Theorem 1.2 without the use of the divergence part of (1.4). Finally, the proof of Proposition 1.3 relies on a result (Lemma 4.1) which is also implicitly contained in Aïdékon and Shi [5] (by following the proof of their Proposition 4.1).

The rest of this paper is organized as follows: In Section 2, we recall some known results on the branching random walk (many-to-one formula, change of measure) and on a real-valued random walk. In Section 3, we prove Theorems 1.1 and 1.2, whereas the proof of Proposition 1.3 will be given in Section 4.

Throughout this paper, $(c_i, 1 \leq i \leq 36)$ denote some positive constants.

2 Preliminaries

2.1 Many-to-one formula for the branching random walk

In this subsection, we recall some change of measure formulas in the branching random walk, for the details we refer to [9, 13, 24, 5, 27] and the references therein.

At first let us fix some notations which will be used throughout this paper: For $|u| = n$, we write $[\varnothing, u] \equiv \{u_0 := \varnothing, u_1, ..., u_{n-1}, u_n = u\}$ the shortest path related u from the root $\varnothing$ such that $|u_i| = i$ for any $0 \leq i \leq n$. For any $u, v \in \mathbb{T}$, we use the partial order $u < v$ if u is an ancestor of v and $u \leq v$ if $u < v$ or $u = v$. We also denote by $\overleftarrow{v}$ the parent of v.

Under (1.1), there exists a centered real-valued random walk $\{S_n, n \geq 0\}$ such that for any $n \geq 1$ and $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbb{E}\Big[\sum_{|u|=n} e^{-V(u)} f(V(u_1), ..., V(u_n)) \Big] = \mathbb{E}\left(f(S_1, ..., S_n) \right). \tag{2.1}$$

Moreover under (1.2), $\sigma^2 = \mathrm{Var}(S_1) = \mathbb{E}\Big[\sum_{|u|=1} (V(u))^2 e^{-V(u)} \Big] \in (0, \infty)$.

The renewal function $R(x)$ related to the random walk S is defined as follows:

$$R(x) := \sum_{k=0}^{\infty} \mathbb{P}\left(S_k \geq -x, \ S_k < \min_{0 \leq j \leq k-1} S_j \right), \qquad x \geq 0, \tag{2.2}$$

and $R(x) = 0$ if $x < 0$. Moreover,

$$\lim_{x \to \infty} \frac{R(x)}{x} = c_R, \tag{2.3}$$

with some positive constant c_R (see Feller [16], pp.612).

For $\alpha \geq 0$, we define as in Aïdékon and Shi [5] two truncated processes: For any $n \geq 0$,

$$W_n^{(\alpha)} := \sum_{|u|=n} e^{-V(u)} 1_{(\underline{V}(u) \geq -\alpha)}, \tag{2.4}$$

$$D_n^{(\alpha)} := \sum_{|u|=n} R_\alpha(V(u)) e^{-V(u)} 1_{(\underline{V}(u) \geq -\alpha))}, \tag{2.5}$$

where $\underline{V}(u) := \min_{\varnothing \leq v \leq u} V(v)$, $R_\alpha(x) := R(\alpha + x)$ and R is the renewal function defined in (2.2).

Denote by $(\mathcal{F}_n, n \geq 0)$ the natural filtration of the branching random walk. If the branching random walk starts from $V(\varnothing) = x$, then we denote its law by $\mathbb{P}_x$ (with $\mathbb{P} = \mathbb{P}_0$). According to Biggins and Kyprianou [9], $(D_n^{(\alpha)}, n \geq 0)$ is a $(\mathbb{P}_x, (\mathcal{F}_n))$-martingale and on some enlarged probability space (more precisely on the space of marked trees enlarged by an infinite ray $(\xi_n, n \geq 0)$, called spine), we may construct a family of probabilities $(\mathbb{Q}_x^{(\alpha)}, x \geq -\alpha)$ such that for any $x \geq -\alpha$, the following statements (i), (ii) and (iii) hold:

(i) For all $n \geq 1$,

$$\frac{d\mathbb{Q}_x^{(\alpha)}}{d\mathbb{P}_x}\Big|_{\mathcal{F}_n} = \frac{D_n^{(\alpha)}}{D_0^{(\alpha)}}, \tag{2.6}$$

$$\mathbb{Q}_x^{(\alpha)}\left(\xi_n = u \big| \mathcal{F}_n\right) = \frac{1}{D_n^{(\alpha)}} R_\alpha(V(u)) e^{-V(u)} 1_{(\underline{V}(u) \geq -\alpha)}, \qquad \forall |u| = n. \tag{2.7}$$

(ii) Under $\mathbb{Q}_x^{(\alpha)}$, the process $\{V(\xi_n), n \geq 0\}$ along the spine $(\xi_n)_{n \geq 0}$, is distributed as the random walk $(S_n, n \geq 0)$ under $\mathbb{P}$ conditioned to stay in $[-\alpha, \infty)$. Moreover for any $n \geq 1, x \geq -\alpha$ and $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbb{E}_{\mathbb{Q}_x^{(\alpha)}}\left[f(V(\xi_1), ..., V(\xi_n))\right] = \frac{1}{R_\alpha(x)} \mathbb{E}_x\left[f(S_1, ..., S_n) R_\alpha(S_n) 1_{(\underline{S}_n \geq -\alpha)}\right]. \tag{2.8}$$

(iii) Let $\mathcal{G}_n := \sigma\{u, V(u) : \overleftarrow{u} \in \{\xi_k, 0 \leq k < n\}\}$, $n \geq 0$. Under $\mathbb{Q}_x^{(\alpha)}$ and conditioned on $\mathcal{G}_\infty$, for all $u \notin \{\xi_k, k \geq 0\}$ but $\overleftarrow{u} \in \{\xi_k, k \geq 0\}$ the induced branching random walk $(V(uv), |v| \geq 0)$ are independent and are distributed as $\mathbb{P}_{V(u)}$, where $\{uv, |v| \geq 0)$ denotes the subtree of $\mathbb{T}$ rooted at u.

Let us mention that as a consequence of (i), the following many-to-one formula holds: For any $n \geq 1, x \geq -\alpha$ and $f : \mathbb{R}^n \to \mathbb{R}_+$,

$$\mathbb{E}_x\left[\sum_{|u|=n} e^{-V(u)} R_\alpha(V(u)) f(V(u_1), ..., V(u_n)) 1_{(\underline{V}(u) \geq -\alpha)}\right] = R(\alpha) e^{-x} \mathbb{E}_{\mathbb{Q}_x^{(\alpha)}}\left[f(V(\xi_1), ..., V(\xi_n))\right]. \tag{2.9}$$

2.2 Estimates on a centered real-valued random walk

We collect here some estimates on a real-valued random walk $\{S_k, k \geq 0\}$, centered and with finite variance $\sigma^2 > 0$. Let $\underline{S}_n := \min_{0 \leq i \leq n} S_i, \forall n \geq 0$. Recall (2.2) for the renewal function $R(\cdot)$.

Fact 2.1 *There exists some constant $c_1 > 0$ such that for any $x \geq 0$,*

$$\mathbb{P}_x\left(\underline{S}_n \geq 0\right) \leq c_1 (1+x) n^{-1/2}, \qquad \forall n \geq 1, \tag{2.10}$$

$$\mathbb{P}_x\left(\underline{S}_{n-1} > S_n \geq 0\right) \leq c_1 (1+x) R(x) n^{-3/2}, \qquad \forall n \geq 1, \tag{2.11}$$

$$\mathbb{P}_x\left(\underline{S}_n \geq 0\right) \sim \theta R(x) n^{-1/2}, \qquad as \ n \to \infty, \tag{2.12}$$

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with $\theta = \frac{1}{c_R}\sqrt{\frac{2}{\pi\sigma^2}}$. Moreover there is $c_2 > 0$ such that for any $b \ge a \ge 0, x \ge 0, n \ge 1$,

$$\mathbb{P}_x\Big(S_n \in [a,b], \underline{S}_n \ge 0\Big) \le c_2(1+x)(1+b-a)(1+b)n^{-3/2}, \tag{2.13}$$

For any $0 < r < 1$, there exists some $c_3 = c_3(r) > 0$ such that for all $b \ge a \ge 0, x, y \ge 0, n \ge 1$,

$$\mathbb{P}_x\Big(S_n \in [y+a, y+b], \underline{S}_n \ge 0, \min_{rn \le j \le n} S_j \ge y\Big) \le c_3\,(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{2.14}$$

See Feller ([16], Theorem 1a, pp.415) for (2.10), Aïdékon and Jaffuel ([4], equation (2.8)) for (2.11), Aïdékon and Shi [5] for (2.13) and (2.14), and Kozlov [22] and Lemma 2.1 in [5] for (2.12) with the identification of the constant $\theta = \frac{1}{c_R}\sqrt{\frac{2}{\pi\sigma^2}}$.

We end this section by an estimate on the stability on x in the convergence (2.12).

Lemma 2.2 *Let S be a centered random walk with positive variance. There exists a constant $c_4 > 0$ such that for all $n \ge 1$ and $x \ge 0$,*

$$\left|\frac{\mathbb{P}_x(\underline{S}_n \ge 0)}{R(x)\mathbb{P}(\underline{S}_n \ge 0)} - 1\right| \le c_4\frac{1+x}{\sqrt{n}}.$$

Proof of Lemma 2.2. Denote in this proof by $\varrho(n) := \mathbb{P}(\underline{S}_n \ge 0)$ for $n \ge 0$. Let $x \ge 0$. By considering the first $k \in [0, n]$ such that $S_k = \underline{S}_n$, we get that

$$
\begin{aligned}
\mathbb{P}_x(\underline{S}_n \ge 0) &= \mathbb{P}_x(\underline{S}_n \ge x) + \sum_{k=1}^n \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0, \min_{k<j\le n} S_j \ge S_k\Big) \\
&= \varrho(n) + \sum_{k=1}^n \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0\Big)\varrho(n-k),
\end{aligned}
$$

by the Markov property at k. Note that $R(x) = 1 + \sum_{k=1}^\infty \mathbb{P}_x\big(\underline{S}_{k-1} > S_k \ge 0\big)$. It follows that

$$\mathbb{P}_x(\underline{S}_n \ge 0) \le R(x)\varrho(n) + \sum_{k=1}^n \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0\Big)[\varrho(n-k) - \varrho(n)]. \tag{2.15}$$

$$\mathbb{P}_x(\underline{S}_n \ge 0) \ge R(x)\varrho(n) - \sum_{k=n+1}^\infty \mathbb{P}_x\Big(\underline{S}_{k-1} > S_k \ge 0\Big)\varrho(n). \tag{2.16}$$

Denote respectively by $I_{(2.15)}$ and $I_{(2.16)}$ the sum $\sum_{k=1}^n$ in (2.15) and the sum $\sum_{k=n+1}^\infty$ in (2.16). Let $T^- := \inf\{j \ge 1 : S_j < 0\}$. By the local limit theorem (Eppel [15], see also [28], equation (22)), if the distribution of S_1 is non-lattice, then

$$\mathbb{P}\Big(T^- = k\Big) \sim \frac{C_-}{k^{3/2}}, \qquad k \to \infty, \tag{2.17}$$

with some positive constant C_-. Moreover Eppel [15] mentioned that a modification of (2.17) holds in the lattice distribution case. Then there exists some constant $c_5 > 0$ such that for all $k \ge 1$,

$$\mathbb{P}\Big(T^- = k\Big) \le \frac{c_5}{k^{3/2}}. \tag{2.18}$$

It follows that for any $k \leq n$, $\varrho(n-k) - \varrho(n) = \mathbb{P}\left(n-k < T^- \leq n\right) \leq c_5 \sum_{i=n-k+1}^{n} i^{-3/2}$. Then by (2.11),

$$I_{(2.15)} \leq c_6(1+x)R(x) \sum_{k=1}^{n} k^{-3/2} \sum_{i=n-k+1}^{n} i^{-3/2}.$$

Elementary computations show that $\sum_{k=1}^{n/2} k^{-3/2} \sum_{i=n-k+1}^{n} i^{-3/2} \leq \sum_{k=1}^{n/2} k^{-3/2} \times k(\frac{n}{2})^{-3/2} = O(\frac{1}{n})$ and $\sum_{k=n/2}^{n} k^{-3/2} \sum_{i=n-k+1}^{n} i^{-3/2} \leq (\frac{n}{2})^{-3/2} \sum_{i=1}^{n} i^{-3/2} \times i = O(\frac{1}{n})$. Hence $I_{(2.15)} \leq c_7(1+x)R(x)\frac{1}{n} \leq c_8(1+x)R(x)\frac{1}{\sqrt{n}}\varrho(n)$ by (2.12).

Finally again by (2.11), we get that $I_{(2.16)} \leq c_9(1+x)R(x)\frac{1}{\sqrt{n}}\varrho(n)$. Then the Lemma follows from (2.15) and (2.16). $\square$

3 Proofs of Theorems 1.1 and 1.2

In view of the inequality: $W_n \geq e^{-\mathbb{M}_n}$, the convergence part of the integral test (1.7) yields that of (1.4), whereas the divergence part of the integral test (1.4) implies that of (1.7). We only need to show the convergence part in (1.7) and the divergence part in (1.4).

3.1 Proof of the convergence part in Theorem 1.2:

Lemma 3.1 *Under* (1.1). *For any* $\alpha \geq 0$, *there exists some constant* $c_{10} = c_{10}(\alpha) > 0$ *such that for any* $1 < n \leq m$ *and* $\lambda > 0$, *we have*

$$\mathbb{P}\left(\max_{n \leq k \leq m} \sqrt{k}W_k^{(\alpha)} > \lambda\right) \leq c_{10}\frac{\log n}{\sqrt{n}} + c_{10}\frac{1}{\lambda}\sqrt{\frac{m}{n}}.$$

Proof of Lemma 3.1. For $n \leq k \leq m+1$, define

$$\widetilde{W}_k^{(\alpha,n)} := \sum_{|u|=k} e^{-V(u)} 1_{(\underline{V}(u_n) \geq -\alpha)},$$

where as before $\underline{V}(u_n) := \min_{1 \leq j \leq n} V(u_j)$ and u_n is the ancestor of u at n-th generation. Then $\widetilde{W}_n^{(\alpha,n)} = W_n^{(\alpha)}$.

For $k \in [n,m]$, $\widetilde{W}_{k+1}^{(\alpha,n)} = \sum_{|v|=k} 1_{(\underline{V}(v_n) \geq -\alpha)} \sum_{u:\overleftarrow{u}=v} e^{-V(u)}$. The branching property implies that $\mathbb{E}\left(\widetilde{W}_{k+1}^{(\alpha,n)}|\mathcal{F}_k\right) = \widetilde{W}_k^{(\alpha,n)}$ for $k \in [n,m]$. By Doob's maximal inequality,

$$\mathbb{P}\left(\max_{n \leq k \leq m} \sqrt{k}\widetilde{W}_k^{(\alpha,n)} \geq \lambda\right) \leq \frac{\sqrt{m}}{\lambda}\mathbb{E}(\widetilde{W}_m^{(\alpha,n)}) = \frac{\sqrt{m}}{\lambda}\mathbb{E}(W_n^{(\alpha)}).$$

By the formula many-to-one (2.1) and the random walk estimate (2.10),

$$\mathbb{E}(W_n^{(\alpha)}) = \mathbb{P}\left(\underline{S}_n \geq -\alpha\right) \leq \frac{c_{11}}{\sqrt{n}},$$

with $c_{11} := c_1(1+\alpha)$. It follows that

$$\mathbb{P}\left(\max_{n \leq k \leq m} \sqrt{k}\widetilde{W}_k^{(\alpha,n)} \geq \lambda\right) \leq \frac{c_{11}}{\lambda}\sqrt{\frac{m}{n}}.$$

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Comparing $\widetilde{W}_k^{(\alpha,n)}$ and $W_k^{(\alpha)}$, we get that

$$\mathbb{P}\Big(\max_{n \leq k \leq m} \sqrt{k} W_k^{(\alpha)} > \lambda \Big) \leq \mathbb{P}\Big(\min_{n \leq k \leq m} \min_{|u|=k} V(u) < -\alpha \Big) + \frac{c_{11}}{\lambda} \sqrt{\frac{m}{n}}.$$

The proof of the Lemma will be finished if we can show that for all $n \geq 2$,

$$\mathbb{P}\Big(\min_{|u| \geq n} V(u) < -\alpha \Big) \leq c_{10} \frac{\log n}{\sqrt{n}}. \tag{3.1}$$

To this end, let us apply the following known result (see e.g. [27]):

$$\mathbb{P}\Big(\inf_{u \in \mathbb{T}} V(u) < -x \Big) \leq e^{-x}, \qquad \forall x \geq 0.$$

Then for all $n \geq 2$,

$$\mathbb{P}\Big(\min_{k \geq n} \min_{|u|=k} V(u) < -\alpha \Big)$$

$$\leq \quad \mathbb{P}\Big(\inf_{u \in \mathbb{T}} V(u) < -\log n \Big) + \mathbb{P}\Big(\min_{k \geq n} \min_{|u|=k} V(u) < -\alpha, \inf_{v \in \mathbb{T}} V(v) \geq -\log n \Big)$$

$$\leq \quad \frac{1}{n} + \sum_{k=n}^{\infty} \mathbb{E}\Big[\sum_{|u|=k} 1_{(V(u) < -\alpha, V(u_n) \geq -\alpha, \dots, V(u_{k-1}) \geq -\alpha, \underline{V}(u) \geq -\log n)} \Big]$$

$$= \quad \frac{1}{n} + \sum_{k=n}^{\infty} \mathbb{E}\Big[e^{S_k} 1_{(S_k < -\alpha, S_n \geq -\alpha, \dots, S_{k-1} \geq -\alpha, \underline{S}_k \geq -\log n)} \Big]$$

$$\leq \quad \frac{1}{n} + e^{-\alpha} \mathbb{P}\Big(\underline{S}_n \geq -\log n \Big),$$

where the above equality is due to the many-to-one formula (2.1). Using (2.10) to bound the above probability term, we get (3.1) and the Lemma. $\square$

Proof of the convergence part in Theorem 1.2: Let f be nondecreasing such that $\int^{\infty} \frac{dt}{t f(t)} < \infty$. Let $n_j := 2^j$ for large $j \geq j_0$. Then $\sum_{j=j_0}^{\infty} \frac{1}{f(n_j)} < \infty$. By using Lemma 3.1,

$$\mathbb{P}\Big(\max_{n_j \leq k \leq n_{j+1}} \sqrt{k} W_k^{(\alpha)} > f(n_j) \Big) \leq c_{10} \frac{\log n_j}{\sqrt{n_j}} + c_{10} \frac{\sqrt{2}}{f(n_j)},$$

whose sum on j converges. The Borel-Cantelli lemma implies that $\mathbb{P}$-a.s. ω, for all large $k \geq k_0(\omega)$, $\sqrt{k} W_k^{(\alpha)} \leq f(k)$. Replacing $f(k)$ by $\varepsilon f(k)$ with an arbitrary constant $\varepsilon > 0$, we get that

$$\limsup_{k \to \infty} \frac{\sqrt{k} W_k^{(\alpha)}}{f(k)} = 0, \qquad \mathbb{P}\text{-a.s.},$$

for any $\alpha \geq 0$. By considering a countable $\alpha \to \infty$ (for instance α integer) and by using the fact that $W_k^{(\alpha)} = W_k$ on the set $\{\inf_{u \in \mathbb{T}} V(u) \geq -\alpha\}$, we get the convergence part. $\square$

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3.2 Proof of the divergence part in Theorem 1.1:

The following lemma is a slight modification of Aïdékon and Shi [5]'s Lemma 6.3:

Lemma 3.2 ([5]) *There exist some constants $K > 0$ and $c_{12} = c_{12}(K) > 0$ such that for all $n \geq 2, 0 \leq \lambda \leq \frac{1}{3} \log n$,*

$$c_{12}\, e^{-\lambda} \leq \mathbb{P}\Big(\bigcup_{k=n+1}^{2n} \big(E_k^{(n,\lambda)} \cap F_k^{(n,\lambda)}\big) \neq \emptyset \Big) \leq \frac{1}{c_{12}} e^{-\lambda}, \tag{3.2}$$

where for $n < k \leq 2n$,

$$E_k^{(n,\lambda)} := \Big\{ u : |u| = k, \frac{1}{2}\log n - \lambda \leq V(u) \leq \frac{1}{2}\log n - \lambda + K,\ V(u_i) \geq a_i^{(n,\lambda)}, \forall 0 \leq i \leq k \Big\},$$

$$F_k^{(n,\lambda)} := \Big\{ u : |u| = k, \sum_{v \in \Upsilon(u_{i+1})} (1 + (V(v) - a_i^{(n,\lambda)})_+)e^{-(V(v) - a_i^{(n,\lambda)})} \leq K\, e^{-b_i^{(k,n)}},\ \forall 0 \leq i < k \Big\},$$

where for $u \in \mathbb{T}\backslash\{\varnothing\}$, $\Upsilon(u) := \{v : v \neq u, \overleftarrow{v} = \overleftarrow{u}\}$ denotes the set of brothers of u, $x_+ := \max(x, 0)$,

$$a_i^{(n,\lambda)} := \Big(\frac{1}{2}\log n - \lambda\Big) 1_{(\frac{n}{2} < i \leq 2n)}, \qquad 0 \leq i \leq 2n,$$

and for $n < k \leq 2n$,

$$b_i^{(k,n)} := i^{1/12} 1_{(0 \leq i \leq \frac{n}{2})} + (k - i)^{1/12}\, 1_{(\frac{n}{2} < i \leq k)}, \qquad 0 \leq i \leq k.$$

Proof of Lemma 3.2. The proof of the lower bound in (3.2) [by the second moment method] goes in the same way as that of Lemma 6.3 in Aïdékon and Shi [5] [We also keep their notations], by replacing $\frac{1}{2}\log n$ in their proof by $\frac{1}{2}\log n - \lambda$. Moreover, a similar computation of the second moment will be given in the proof of Lemma 3.3. Then we omit the details.

The upper bound in (3.2) is a simple consequence of the many-to-one formula: Define $s := \frac{1}{2}\log n - \lambda$, we have that

$$\mathbb{P}\Big(\bigcup_{k=n+1}^{2n} E_k^{(n,\lambda)} \neq \emptyset \Big) \leq \sum_{k=n+1}^{2n} \mathbb{E}\Big[\sum_{|u|=k} 1_{(s \leq V(u) \leq s+K, V(u_i) \geq a_i^{(n,\lambda)}, \forall i \leq k)} \Big]$$

$$= \sum_{k=n+1}^{2n} \mathbb{E}\Big[e^{S_k} 1_{(s \leq S_k \leq s+K, S_i \geq a_i^{(n,\lambda)}, \forall i \leq k)} \Big]$$

$$\leq \sum_{k=n+1}^{2n} e^{s+K}\, \mathbb{P}\Big(s \leq S_k \leq s + K, S_i \geq a_i^{(n,\lambda)}, \forall i \leq k \Big).$$

By (2.14), $\mathbb{P}\big(s \leq S_k \leq s + K, S_i \geq a_i^{(n,\lambda)}, \forall i \leq k\big) \leq c_{13} n^{-3/2}$ for all $n < k \leq 2n$. Hence $\mathbb{P}\big(\bigcup_{k=n+1}^{2n} E_k^{(n,\lambda)} \neq \emptyset\big) \leq c_{13} e^{-\lambda+K}$ proving the upper bound in (3.2). $\square$

Using the notations in Lemma 3.2 with the constant K, we define for $n \geq 2$ and $0 \leq \lambda \leq \frac{1}{3}\log n$,

$$A(n, \lambda) := \Big\{ \cup_{k=n+1}^{2n} \big(E_k^{(n,\lambda)} \cap F_k^{(n,\lambda)}\big) \neq \emptyset \Big\}. \tag{3.3}$$

The following estimate will be useful in the application of Borel-Cantelli's lemma:

Lemma 3.3 *There exists some constant $c_{14} > 0$ such that for any $n \geq 2, 0 \leq \lambda \leq \frac{1}{3}\log n$ and $m \geq 4n, 0 \leq \mu \leq \frac{1}{3}\log m$,*

$$\mathbb{P}\Big(A(n,\lambda) \cap A(m,\mu)\Big) \leq c_{14}\, e^{-\lambda-\mu} + c_{14}\, e^{-\mu}\frac{\log n}{\sqrt{n}}.$$

Proof of Lemma 3.3. As we mentioned before, the arguments that we use are very close to the computation of the second moment in the proof of Lemma 6.3 in [5]. Write for brevity in this proof

$$s := \frac{1}{2}\log n - \lambda, \qquad t := \frac{1}{2}\log m - \mu.$$

Similarly to (2.6) and (2.7), we may construct a new probability $\mathbb{Q}$ such that for all $n \geq 1$, $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_n} = W_n$, $\mathbb{Q}\big(\xi_n = u \big| \mathcal{F}_n\big) = \frac{e^{-V(u)}}{W_n}, \forall |u| = n$. Moreover under $\mathbb{Q}$, $(V(\xi_n), n \geq 0)$ is distributed as the random walk $(S_n, n \geq 0)$ defined in Section 2, and the spine decomposition similar to (iii) in Section 2 holds under $\mathbb{Q}$. We refer to [9, 13, 24, 5, 27] for details. It follows that

$$
\begin{aligned}
\mathbb{P}\Big(A(n,\lambda) \cap A(m,\mu)\Big) &\leq \mathbb{E}\Big[1_{A(n,\lambda)} \sum_{k=m+1}^{2m} \sum_{|u|=k} 1_{(u \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})}\Big] \\
&= \sum_{k=m+1}^{2m} \mathbb{E}_{\mathbb{Q}}\Big[1_{A(n,\lambda)} e^{V(\xi_k)} 1_{(\xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})}\Big] \\
&\leq e^{t+K} \sum_{k=m+1}^{2m} \mathbb{E}_{\mathbb{Q}}\Big[A(n,\lambda),\, \xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)}\Big] \\
&\leq e^{t+K} \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} \mathbb{E}_{\mathbb{Q}}\Big[\sum_{|v|=l} 1_{(v \in E_l^{(n,\lambda)} \cap F_l^{(n,\lambda)},\, \xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})}\Big] \\
&=: e^{t+K} \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} I_{(3.4)}(k,l). \quad (3.4)
\end{aligned}
$$

For $n < l \leq 2n \leq \frac{m}{2} < k \leq 2m$, we may decompose the sum on $|v| = l$ as follows:

$$\sum_{|v|=l} 1_{(v \in E_l^{(n,\lambda)} \cap F_l^{(n,\lambda)})} = 1_{(\xi_l \in E_l^{(n,\lambda)} \cap F_l^{(n,\lambda)})} + \sum_{p=1}^{l} \sum_{u \in \Upsilon(\xi_p)} \sum_{v \in \mathbb{T}(u), |v|_u = l-p} 1_{(v \in E_l^{(n,\lambda)} \cap F_l^{(n,\lambda)})},$$

where $\mathbb{T}(u)$ denotes the subtree of $\mathbb{T}$ rooted at u and $|v|_u = |v| - |u|$ the relative generation of $v \in \mathbb{T}(u)$. Then

$$
\begin{aligned}
&I_{(3.4)}(k,l) \\
&= \mathbb{Q}\Big(\xi_l \in E_l^{(n,\lambda)} \cap F_l^{(n,\lambda)}, \xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)}\Big) + \sum_{p=1}^{l} \mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k \in E_k^{(m,\mu)} \cap F_k^{(m,\mu)})} \sum_{u \in \Upsilon(\xi_p)} f_{k,l,p}(V(u))\Big] \\
&=: I_{(3.5)}(k,l) + \sum_{p=1}^{l} J_{(3.5)}(k,l,p), \quad (3.5)
\end{aligned}
$$

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with
$$f_{k,l,p}(x) := \mathbb{E}_{\mathbb{Q}}\Big[\sum_{v\in\mathbb{T}(u),|v|_u=l-p} 1_{(v\in E_l^{(n,\lambda)}\cap F_l^{(n,\lambda)})}\big|V(u)=x\Big], \qquad x\in\mathbb{R}.$$

In what follows, we shall at first estimate $J_{(3.5)}(k,l,p)$ then $I_{(3.5)}(k,l)$. By the branching property at u and by removing the event $F_l^{(n,\lambda)}$ from the indicator function in $f_{k,l,p}(r)$, we get that

$$
\begin{aligned}
f_{k,l,p}(x) &\leq \mathbb{E}_x\Big[\sum_{|v|=l-p} 1_{(s\leq V(v)\leq s+K, V(v_i)\geq a_{i+p}^{(n,\lambda)}, \forall 0\leq i\leq l-p)}\Big] \\
&= e^{-x}\mathbb{E}_x\Big[e^{S_{l-p}} 1_{(s\leq S_{l-p}\leq s+K, S_i\geq a_{i+p}^{(n,\lambda)}, \forall 0\leq i\leq l-p)}\Big] \\
&\leq e^{-x+s+K}\mathbb{P}_x\Big(s\leq S_{l-p}\leq s+K, S_i\geq a_{i+p}^{(n,\lambda)}, \forall 0\leq i\leq l-p\Big),
\end{aligned}
\tag{3.6}
$$

where to get the above equality, we applied an obvious modification of (2.1) for $\mathbb{E}_x$ instead of $\mathbb{E}$.

Let us denote by $(3.6)_{k,l,p}$ the probability term in (3.6). To estimate $(3.6)_{k,l,p}$, we distinguish as in [5] two cases: $p\leq\frac{n}{2}$ and $\frac{n}{2}<p\leq l$. Recall that $n<l\leq 2n\leq\frac{m}{2}<k\leq 2m$. If $p\leq\frac{n}{2}$,

$$(3.6)_{k,l,p}\leq 1_{(x\geq 0)}c_{15}\frac{1+x}{(l-p)^{3/2}},$$

by using (2.14). Then for $1\leq p\leq\frac{n}{2}$,

$$f_{k,l,p}(x)\leq c_{15}1_{(x\geq 0)}e^{s+K-x}(1+x)(l-p)^{-3/2}.$$

It follows that for all $n<l\leq 2n, m<k\leq 2m$,

$$
\begin{aligned}
\sum_{1\leq p\leq n/2} J_{(3.5)}(k,l,p) &\leq \sum_{p=1}^{n/2}\mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k\in E_k^{(m,\mu)}\cap F_k^{(m,\mu)})}\sum_{u\in\Upsilon(\xi_p)} c_{15}1_{(V(u)\geq 0)}e^{s+K-V(u)}\frac{1+V(u)}{(l-p)^{3/2}}\Big] \\
&\leq c_{16}\,e^s n^{-3/2}\sum_{p=1}^{n/2}\mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k\in E_k^{(m,\mu)}\cap F_k^{(m,\mu)})}\sum_{u\in\Upsilon(\xi_p)} 1_{(V(u)\geq 0)}e^{-V(u)}(1+V(u))\Big] \\
&\leq c_{16}\,K\,e^s n^{-3/2}\sum_{p=1}^{n/2}\mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k\in E_k^{(m,\mu)}\cap F_k^{(m,\mu)})}e^{-(p-1)^{1/12}}\Big],
\end{aligned}
$$

where the last inequality is due to the definition of $\xi_k\in F_k^{(m,\mu)}$ [noticing that $a_p^{(m,\mu)}=0$ and $b_p^{(k,m)}=p^{1/12}$ for all $p\leq n/2<m/2$]. Then we get that

$$\sum_{1\leq p\leq n/2} J_{(3.5)}(k,l,p)\leq c_{17}e^s n^{-3/2}\,\mathbb{Q}\Big(\xi_k\in E_k^{(m,\mu)}\Big)\leq c_{18}e^s n^{-3/2}\,m^{-3/2}, \tag{3.7}$$

since $\mathbb{Q}\Big(\xi_k\in E_k^{(m,\mu)}\Big)=\mathbb{P}(t\leq S_k\leq t+K, S_i\geq a_i^{(m,\mu)}, \forall 0\leq i\leq k)\leq c_{19}m^{-3/2}$ for all $m<k\leq 2m$, by using (2.14).

Now considering $\frac{n}{2}<p\leq l$, $a_{i+p}^{(n,\lambda)}=s$ for any $0\leq i\leq l-p$, hence

$$(3.6)_{k,l,p}=1_{(x\geq s)}\mathbb{P}_x\Big(s\leq S_{l-p}\leq s+K, \underline{S}_{l-p}\geq s\Big)\leq 1_{(x\geq s)}c_2(1+K)^2\frac{1+x-s}{(1+l-p)^{3/2}},$$

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by (2.13). It follows that

$$\sum_{\frac{n}{2}<p\leq l} J_{(3.5)}(k,l,p) \leq \sum_{\frac{n}{2}<p\leq l} \mathbb{E}_{\mathbb{Q}}\Big[1_{(\xi_k\in E_k^{(m,\mu)}\cap F_k^{(m,\mu)})}\sum_{u\in\Upsilon(\xi_p)} c_2(1+K)^2 1_{(V(u)\geq s)}e^{s+K-V(u)}\frac{1+V(u)-s}{(1+l-p)^{3/2}}\Big].$$

By the definition of $\xi_k\in F_k^{(m,\mu)}$, for all $p\leq l\leq 2n\leq\frac{m}{2}$, we have that

$$\sum_{u\in\Upsilon(\xi_p)} 1_{(V(u)\geq s)}e^{-V(u)}(1+V(u)-s) \leq \sum_{u\in\Upsilon(\xi_p)} 1_{(V(u)\geq 0)}e^{-V(u)}(1+V(u)) \leq Ke^{-(p-1)^{1/12}}.$$

Then

$$\sum_{\frac{n}{2}\leq p\leq l} J_{(3.5)}(k,l,p) \leq c_{20}e^s\,e^{-n^{1/13}}\mathbb{Q}(\xi_k\in E_k^{(m,\mu)}) \leq c_{21}e^{s-n^{1/13}}m^{-3/2}. \tag{3.8}$$

Combining (3.7) and (3.8), we get that

$$\sum_{1\leq p\leq l} J_{(3.5)}(k,l,p) \leq (c_{18}n^{-3/2}+c_{21}e^{-n^{1/13}})e^s\,m^{-3/2}. \tag{3.9}$$

It remains to estimate $I_{(3.5)}(k,l)$ for $n<l\leq 2n$ and $m<k\leq 2m$ [in particular $l<k$]. We have

$$I_{(3.5)}(k,l)$$
$$\leq \mathbb{Q}\Big(\xi_l\in E_l^{(n,\lambda)},\xi_k\in E_k^{(m,\mu)}\Big)$$
$$= \mathbb{P}\Big(s\leq S_l\leq s+K, S_i\geq a_i^{(n,\lambda)},\forall i\leq l, t\leq S_k\leq t+K, S_j\geq a_j^{(m,\mu)},\forall j\leq k\Big). \tag{3.10}$$

Let us denote by $(3.10)_{k,l}$ the probability term in (3.10). Using the Markov property at l, we get that

$$(3.10)_{k,l} = \mathbb{E}\Big[1_{(s\leq S_l\leq s+K,S_i\geq a_i^{(n,\lambda)},\forall 0\leq i\leq l)}\mathbb{P}_{S_l}\Big(t\leq S_{k-l}\leq t+K, S_j\geq a_{j+l}^{(m,\mu)},\forall 0\leq j\leq k-l\Big)\Big]$$
$$\leq \frac{c_{21}}{(k-l)^{3/2}}\mathbb{E}\Big[1_{(s\leq S_l\leq s+K,S_i\geq a_i^{(n,\lambda)},\forall 0\leq i\leq l)}(1+S_l)\Big] \qquad \text{(by (2.13))}$$
$$\leq c_{22}(1+s+K)(k-l)^{-3/2}\,l^{-3/2}.$$

Based on the above estimate and (3.9), we deduce from (3.4) and (3.5) that

$$\mathbb{P}\Big(A(n,\lambda)\cap A(m,\mu)\Big)$$
$$\leq c_{23}\,e^{t+K}\sum_{k=m+1}^{2m}\sum_{l=n+1}^{2n}\Big((1+s+K)(k-l)^{-3/2}\,l^{-3/2}+e^s\,e^{-n^{1/13}}m^{-3/2}+e^s n^{-3/2}\,m^{-3/2}\Big)$$
$$\leq c_{24}e^{-\lambda-\mu}+c_{24}e^{-\mu}\frac{\log n}{\sqrt{n}},$$

proving the Lemma. $\square$

Proof of the divergence part in Theorem 1.1. Let f be nondecreasing such that $\int^\infty \frac{dt}{te^{f(t)}}=\infty$. Without any loss of generality we may assume that $\sqrt{\log t}\leq e^{f(t)}\leq(\log t)^2$ for all large $t\geq t_0$ (see e.g. [14] for a similar justification). Let us prove that there exists some $c_{25}>0$ such that for any $x\in\mathbb{R}$ and $k\geq 0$,

$$\mathbb{P}\Big(\mathbb{M}_n+x\leq\frac{1}{2}\log n-f(n+k),\text{ i.o. }n\to\infty\Big)\geq c_{25}. \tag{3.11}$$

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To this end, we take $n_i := 2^i$ for $i \geq 1$, $\lambda_i := f(n_i + k) + x + K$ and consider the event $A_i := A(n_i, \lambda_i)$ in (3.3). By Lemma 3.2,

$$c_{12} e^{-\lambda_i} \leq \mathbb{P}(A_i) \leq \frac{1}{c_{12}} e^{-\lambda_i}.$$

Note that $\sum_i e^{-\lambda_i} = \infty$. By Lemma 3.3, we have for any $j \geq i + 2$,

$$\mathbb{P}\Big(A_i \cap A_j\Big) \leq c_{14} e^{-\lambda_i - \lambda_j} + c_{14} e^{-\lambda_j} \frac{\log n_i}{\sqrt{n_i}}.$$

It follows that

$$\limsup_{k \to \infty} \frac{\sum_{1 \leq i,j \leq k} \mathbb{P}(A_i \cap A_j)}{\left[\sum_{i=1}^k \mathbb{P}(A_i)\right]^2} \leq \frac{c_{14}}{c_{12}^2},$$

which does not depend on (x, k). By Kochen and Stone [21]'s version of the Borel-Cantelli lemma, $\mathbb{P}(A_i, \text{i.o. } i \to \infty) \geq c_{12}^2/c_{14} =: c_{25}$, hence we get (3.11).

Denote by

$$B_x(k) := \Big\{\mathbb{M}_n + x \leq \frac{1}{2} \log n - f(n + k), \text{ i.o.}\Big\}, \qquad x \in \mathbb{R}, \ k \geq 0.$$

We have proved that for any $x \in \mathbb{R}$ and $k \geq 0$, $\mathbb{P}(B_x(k)) \geq c_{25}$. For any $k \geq 0$, the events $B_x(k)$ are non-increasing on x. Let $B_\infty(k) := \cap_{i=1}^\infty B_i(k)$ [then $B_\infty(k)$ is the event nothing but $\{\liminf_{n \to \infty}(\mathbb{M}_n - \frac{1}{2} \log n + f(n + k)) = -\infty\}$]. By the monotone convergence, $\mathbb{P}(B_\infty(k)) \geq c_{25}$, for all $k \geq 0$. Moreover, for any $x \in \mathbb{R}$, $\mathbb{P}_x(B_\infty(k)) = \mathbb{P}(B_\infty(k)) \geq c_{25}$. On the other hand, if we denote by $Z_k := \sum_{|u|=k} 1$ the number of particles in the k-th generation, then by the branching property,

$$\mathbb{P}\Big(B_\infty(0) \,\big|\, \mathcal{F}_k\Big) = 1_{(Z_k > 0)}\Big(1 - \prod_{|u|=k}(1 - \mathbb{P}_{V(u)}(B_\infty(k)))\Big) \geq 1_{(Z_k > 0)}\Big(1 - (1 - c_{25})^{Z_k}\Big).$$

It is well-known (cf. [7], pp.8) that on $\{Z_k > 0\}$, $Z_k \to \infty$ and $1_{(Z_k > 0)} \to 1_{\mathcal{S}}$ (in probability), hence $\mathbb{P}$-a.s.,

$$1_{B_\infty(0)} = \lim_{k \to \infty} \mathbb{P}\Big(B_\infty(0) \,\big|\, \mathcal{F}_k\Big) \geq 1_{\mathcal{S}}.$$

Clearly $\mathcal{S}^c \subset B_\infty(0)^c$ by the convention on the definition of $\mathbb{M}_n$ on $\mathcal{S}^c$. Hence $\mathbb{P}$-a.s. on $\mathcal{S} = B_\infty(0)$. This proves the divergence part of Theorem 1.1. $\square$

4 Proof of Proposition 1.3

The main technical part was already done in Aïdékon and Shi [5]:

Lemma 4.1 ([5]) *Under* (1.1) *and* (1.8). *For any fixed* $\alpha \geq 0$, *there exist some* $\delta = \delta(\varepsilon_0) > 0$ *and* $c_{26} = c_{26}(\alpha, \delta) > 0$ *such that for all* $n \geq 2$,

$$Var_{\mathbb{Q}^{(\alpha)}}\left(\frac{\sqrt{n} W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) \leq c_{26}\left(n^{-\delta} + \sup_{k_n^{1/3} \leq x \leq k_n} \left|\frac{h_{x+\alpha}(n - k_n)}{h_\alpha(n)} - 1\right|\right), \tag{4.1}$$

where $k_n := \lfloor n^{1/3} \rfloor$ *and* $h_x(j) := \frac{\sqrt{j} \, \mathbb{P}_x(S_j \geq 0)}{R(x)}$ *for* $j \geq 1$, $x \geq 0$.

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Proof of Lemma 4.1. The Lemma was implicitly contained in the proof of Proposition 4.1 in [5]. In fact, in their proof of the convergence that $\mathrm{Var}_{\mathbb{Q}^{(\alpha)}}\left(\frac{\sqrt{n}W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) \to 0$, we choose $k_n := \lfloor n^{1/3} \rfloor$ in their definition of $E_n := E_{n,1} \cap E_{n,2} \cap E_{n,3}$ (see the equation (4.6) in [5], Section 4). We claim that for some constant $\delta_1 = \delta_1(\varepsilon_0) > 0$, there is some $c_{27} = c_{27}(\delta_1, \alpha) > 0$ such that for all $n \geq 1$,

$$\mathbb{Q}^{(\alpha)}\left(E_n^c\right) \leq c_{27}\, n^{-\delta_1}, \tag{4.2}$$

$$\sup_{k_n^{1/3} \leq x \leq k_n} \mathbb{Q}^{(\alpha)}\left(E_n^c \big| V(\xi_{k_n}) = x\right) \leq c_{27} n^{-\delta_1}. \tag{4.3}$$

In fact, according to the definition of $E_{n,1}$ in [5],

$$\mathbb{Q}^{(\alpha)}\left(E_{n,1}^c\right) \leq \mathbb{Q}^{(\alpha)}\left(\{V(\xi_{k_n}) > k_n\} \cup \{V(\xi_{k_n}) < k_n^{1/3}\}\right) + \sup_{k_n^{1/3} \leq x \leq k_n} \mathbb{Q}_x^{(\alpha)}\left(\cup_{i=0}^{n-k_n}\{V(\xi_i) < k_n^{1/6}\}\right).$$

By (2.8),

$$\mathbb{Q}^{(\alpha)}\left(V(\xi_{k_n}) < k_n^{1/3}\right) = \frac{1}{R_\alpha(0)}\mathbb{E}\left(1_{(\underline{S}_{k_n} \geq -\alpha,\, S_{k_n} < k_n^{1/3})} R_\alpha(S_{k_n})\right) \leq \frac{R_\alpha(k_n^{1/3})}{R_\alpha(0)} \mathbb{P}(\underline{S}_{k_n} \geq -\alpha) \leq c_{28} k_n^{-1/6},$$

and

$$\mathbb{Q}^{(\alpha)}\left(V(\xi_{k_n}) > k_n\right) \leq \mathbb{E}\left[1_{(S_{k_n} > k_n)} R_\alpha(S_{k_n})\right] \leq \sqrt{\mathbb{P}\left(S_{k_n} > k_n\right)}\sqrt{\mathbb{E}[R_\alpha(S_{k_n})^2]} \leq c_{29} k_n \sqrt{\mathbb{P}\left(S_{k_n} > k_n\right)},$$

since $R_\alpha(x) \sim c_R(1 + x + \alpha)$ as $x \to \infty$. The condition (1.8) ensures that $\mathbb{E}(|S_1|^{2+\varepsilon_0}) < \infty$ which in turn implies that $\mathbb{E}(|S_k|^{2+\varepsilon_0}) \leq c_{30} k^{1+\varepsilon_0/2}$ for any $k \geq 1$ (see Petrov [26], pp.60). Hence

$$\mathbb{Q}^{(\alpha)}\left(V(\xi_{k_n}) > k_n\right) \leq c_{31} k_n^{-\varepsilon_0/4}.$$

Now for $k_n^{1/3} \leq x \leq k_n$, let $\tau = \inf\{i \geq 0 : S_i < k_n^{1/6}\}$, then the absolu continuity (2.8) at τ reads as

$$\mathbb{Q}_x^{(\alpha)}\left(\cup_{i=0}^{n-k_n}\{V(\xi_i) < k_n^{1/6}\}\right) = \frac{1}{R_\alpha(x)}\mathbb{E}_x\left[1_{(\tau \leq n-k_n)} R_\alpha(S_\tau) 1_{(\underline{S}_\tau \geq -\alpha)}\right]$$

$$\leq \frac{R_\alpha(k_n^{1/6})}{R_\alpha(x)} \leq \frac{R_\alpha(k_n^{1/6})}{R_\alpha(k_n^{1/3})} \leq c_{32}\, k_n^{-1/6},$$

since $x \geq k_n^{1/3}$. Assembling the above estimates yields that

$$\mathbb{Q}^{(\alpha)}\left(E_{n,1}^c\right) \leq c_{33} k_n^{-\varepsilon_0/4},$$

[we may assume $\varepsilon_0 \leq 2/3$]. Let us follow the proof of Lemma 4.7 in [5], we remark that on $E_{n,1}$, $V(\xi_i) \geq k_n^{1/6}$ for all $k_n \leq i \leq n$, and it was shown in [5] that

$$\mathbb{Q}_x^{(\alpha)}\left(E_{n,1} \cap E_{n,2}^c\right) \leq \sum_{i=k_n}^{n-1} \mathbb{E}_x\left[1_{(\eta+\widetilde{\eta} > e^{S_i/2})}\left[\eta + \frac{\widetilde{\eta}}{S_i + \alpha + 1}\right] 1_{(S_i \geq k_n^{1/6})}\right].$$

By the integrability assumption (1.8), since $\widetilde{\eta} = \int_0^\infty x e^{-x} \mathcal{L}(dx) \leq \sqrt{\eta \int_0^\infty x^2 e^{-x} \mathcal{L}(dx)}$, it is easy to see that $\mathbb{E}(\widetilde{\eta}^p) < \infty$ for some $p > 1$. It follows that $\mathbb{Q}_x^{(\alpha)}\left(E_{n,1} \cap E_{n,2}^c\right) \leq n\,\mathbb{E}\left(1_{(\eta+\widetilde{\eta}>e^{k_n^{1/6}/2})}[\eta + \widetilde{\eta}]\right) \leq c_{34}n^{-10}$. Finally by (4.9) in [5], $\mathbb{Q}^{(\alpha)}(E_{n,1} \cap E_{n,2} \cap E_{n,3}^c) \leq c_{35}n^{-10}$, hence we get (4.2).

From (4.2), it suffices to follow line-by-line the proof of Proposition 4.1 in [5]: In Lemma 4.4 of [5], we can get $n^{-1-\delta_1/4}$ instead of $o(\frac{1}{n})$ [by replacing in its proof ε by $n^{-\delta_1/4}$]. In their proof of Lemma 4.5, taking $\eta_1 = \frac{1}{n}$ and we arrive at

$$\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\left(\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right)\right]^2 \leq c_{36}n^{-\delta_1/4} + (1+O(\tfrac{1}{n}))\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right] \sup_{k_n^{1/3}\leq x\leq k_n} \frac{\sqrt{n}\mathbb{P}(\underline{S}_{n-k_n} \geq -\alpha - x)}{R_\alpha(x)}.$$

The Lemma follows because $\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right] = h_\alpha(n)$, and $h_\alpha(n) \to \theta$ when $n \to \infty$, as shown in [5]. $\square$

Proof of Proposition 1.3. It is enough to prove that for any $\alpha \geq 0$,

$$\liminf_{n\to\infty} \sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} = \theta, \qquad \mathbb{Q}^{(\alpha)}\text{-a.s.,} \tag{4.4}$$

where θ is defined in (2.12). In fact, under (1.1), (1.2) and (1.3), $D_n^{(\alpha)}$ converges in mean to $D_\infty^{(\alpha)}$ (see [27], Chapter 5, also see [9], Theorem 10.2 (i) with an extra $\log\log\log$-term). Then on $\{D_\infty^{(\alpha)} > 0\}$, $\mathbb{P}$ and $\mathbb{Q}^{(\alpha)}$ are equivalent. Moreover, as shown in [5], $\mathbb{P}$-almost surely on $\{\inf_{|u|\geq 0} V(u) \geq -\alpha\}$, $W_n^{(\alpha)} = W_n$ and $D_n^{(\alpha)} \sim c_R D_\infty$, therefore Proposition 1.3 follows easily from (4.4).

Now to prove (4.4), since $\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \to \theta$ in probability under $\mathbb{Q}^{(\alpha)}$ ([5]), it suffices to prove that

$$\liminf_{n\to\infty} \sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \geq \theta, \qquad \mathbb{Q}^{(\alpha)}\text{-a.s.} \tag{4.5}$$

To this end, using Lemmas 4.1 and 2.2 we get some constant $\delta_2 > 0$ such that for all $n \geq n_0$,

$$\mathrm{Var}_{\mathbb{Q}^{(\alpha)}}\left(\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}}\right) \leq n^{-\delta_2}. \tag{4.6}$$

Let $n_j := j^{-3/\delta_2}$ for $j \geq j_0$ and choose an arbitrary small $\varepsilon > 0$. We are going to show that

$$\sum_{j\geq j_0} \mathbb{Q}^{(\alpha)}\left(\inf_{n_j \leq n \leq n_{j+1}} \sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} < (1-\varepsilon)\theta\right) < \infty, \tag{4.7}$$

from which the Borel-Cantelli lemma yields (4.5).

To prove (4.7), let $\widehat{\mathcal{F}}_n := \mathcal{F}_n \vee \mathcal{G}_n$, where $\mathcal{G}_n$, defined in Section 2, denotes the σ-fields generated by the spine up to generation n. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\frac{1}{R_\alpha(V(\xi_{n_{j+1}}))}\Big|\,\widehat{F}_n\right] &= \mathbb{E}_{\mathbb{Q}_{S_n}^{(\alpha)}}\left[\frac{1}{R_\alpha(V(\xi_{n_{j+1}-n}))}\right] \\
&= \frac{1}{R_\alpha(V(\xi_n))}\mathbb{P}\left(\underline{S}_{n_{j+1}-n} \geq -\alpha\right) \qquad \text{(by (2.8))} \\
&\leq \frac{1}{R_\alpha(V(\xi_n))}.
\end{aligned}$$

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It follows that for all $n \leq n_{j+1}$,

$$\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\frac{1}{R_\alpha(V(\xi_{n_{j+1}}))}\Big|\mathcal{F}_n\right] \leq \mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\frac{1}{R_\alpha(V(\xi_n))}\Big|\mathcal{F}_n\right] = \frac{W_n^{(\alpha)}}{D_n^{(\alpha)}},$$

where the last equality comes from Lemma 4.2 in [5]. Consequently for all $n_j \leq n \leq n_{j+1}$,

$$\sqrt{n}\,\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} \geq Y_n := \sqrt{n_j}\,\mathbb{E}_{\mathbb{Q}^{(\alpha)}}\left[\frac{1}{R_\alpha(V(\xi_{n_{j+1}}))}\Big|\mathcal{F}_n\right].$$

Remark that $(Y_n, n_j \leq n_{j+1})$ is a martingale with mean $\mathbb{E}_{\mathbb{Q}^{(\alpha)}}(Y_{n_j}) = \sqrt{n_j}\,\mathbb{E}_{\mathbb{Q}^{(\alpha)}}(\frac{1}{R_\alpha(S_{n_j})}) \geq (1-\varepsilon)\theta$. The Doob L^2-inequality implies that

$$\begin{aligned}
\mathbb{Q}^{(\alpha)}\left(\max_{n_j \leq n \leq n_{j+1}}\left|Y_n - \mathbb{E}_{\mathbb{Q}^{(\alpha)}}(Y_{n_j})\right| \geq \varepsilon\theta/2\right) &\leq \frac{4}{\varepsilon^2\theta^2}\mathrm{Var}_{\mathbb{Q}^{(\alpha)}}(Y_{n_{j+1}}) \\
&\leq c_{36}\,n_j^{-\delta_2} = c_{36}j^{-3},
\end{aligned}$$

by (4.6) and the fact that $Y_{n_{j+1}} = \sqrt{\frac{n_j}{n_{j+1}}}\,\frac{W_{n_{j+1}}^{(\alpha)}}{D_{n_{j+1}}^{(\alpha)}}$. Finally for all large j,

$$\mathbb{Q}^{(\alpha)}\left(\inf_{n_j \leq n \leq n_{j+1}}\sqrt{n}\frac{W_n^{(\alpha)}}{D_n^{(\alpha)}} < (1-\varepsilon)\theta\right) \leq \mathbb{Q}^{(\alpha)}\left(\max_{n_j \leq n \leq n_{j+1}}\left|Y_n - \mathbb{E}_{\mathbb{Q}^{(\alpha)}}(Y_{n_j})\right| \geq \varepsilon\theta/2\right) \leq c_{36}\,j^{-3},$$

proving (4.7) and then completing the proof of Proposition 1.3. $\square$

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